SERIES REPRESENTATION OF GENERALIZED TEMPERATURE FUNCTIONS*

By

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1. Introduction:

In a recent paper [6], the author established criteria for the expansion of a solution of the generalized heat equation

$$(1.1) \qquad \frac{\partial^2}{\partial x^2} u(x,t) + \frac{2\nu}{x} \frac{\partial}{\partial x} u(x,t) = \frac{\partial}{\partial t} u(x,t) ,$$

ν a fixed positive number, in a series of polynomial solutions $P_{n,\nu}(x,t)$, where

$$(1.2) \quad P_{n,\nu}(x,t) = \sum_{k=0}^{n} 2^{2k} \binom{n}{k} \frac{\Gamma(\nu+\frac{1}{2}+n)}{\Gamma(\nu+\frac{1}{2}+n-k)} x^{2n-2k} t^k .$$

See also [1]. It is our present goal to establish that these same criteria are necessary and sufficient for the representation of a solution of (1.1) by a Maclaurin series in x^2 and t. Analogous theorems for the ordinary heat equation are derived in [10].

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For computational purposes, these results provide a useful means of evaluating those solutions which have Maclaurin series expansions. In the case of the radial flow of heat in an infinite circular cylinder, for example, the temperature satisfies the equation

$$(1.3) \qquad \frac{\partial^2 u}{\partial x^2}(x,t) + \frac{1}{x}\frac{\partial u}{\partial t}(x,t) = \frac{\partial u}{\partial t}(x,t), \quad 0 < x < a .$$

If the surface $x = a$ is kept at zero temperature so that

$$(1.4) \qquad u(a,t) = 0 .$$

and if the initial temperature is given by

$$(1.5) \qquad u(x,0) = f(x) ,$$

then

$$(1.6) \qquad u(x,t) = \frac{2}{a^2} \sum_{n=1}^{\infty} e^{-\alpha_n^2 t} \frac{J_o(\alpha_n x)}{J_1^2(\alpha_n a)} \int_0^a u\, f(u)\, J_o(u\alpha_n)\, du ,$$

where $J_o(\alpha_n a) = 0$, $n = 1, 2, \ldots$. See, for example, [2;7.4-7.6]. Since $u(x,t)$ can be shown to satisfy the criteria of our theory, it may also be expressed in the simpler form of a double Maclaurin series.

2. Definitions and preliminary results:

The fundamental solution of the generalized heat equation (1.1) is the function $G(x,y;t)$ given by

$$(2.1)\qquad G(x,y;t) = \left(\frac{1}{2t}\right)^{\nu+\frac{1}{2}} e^{-\frac{x^2+y^2}{4t}} \mathcal{J}\left(\frac{xy}{2t}\right),$$

where

$$\mathcal{J}(z) = 2^{\nu-\frac{1}{2}}\,\Gamma(\nu+\tfrac{1}{2})\, z^{\frac{1}{2}-\nu}\, I_{\nu-\frac{1}{2}}(z),$$

$I_\alpha(z)$ being the Bessel function of imaginary argument of order α. By a direct computation, we may establish each of the following two results.

LEMMA 2.1. For $0 \leq x,\ y < \infty,\ t > 0$,

$$(2.2)\qquad \left(\frac{\partial}{\partial x}\right)^{2n} G(x,y;t)\Big|_{x=0}$$

$$= \frac{(2n)!\,\Gamma(\nu+\frac{1}{2})}{2^{4n}\, n!\,\Gamma(\nu+\frac{1}{2}+n)}\, t^{-2n}\, G(y;t)\, P_{n,\nu}(y,-t),$$

where the $P_{n,\nu}(n,t)$ are given by (1.2).

LEMMA 2.2. For $0 \leq x,\ y < \infty,\ |t| < t',\ t' > 0$,

$$(2.3)\qquad G(x,y;\, t+t')$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!\,\Gamma(\nu+\frac{1}{2}+n)} \sum_{m=0}^{\infty} \frac{2^{2m} t^m}{m!}\left[\frac{G(y;t')\,\Gamma(\nu+\frac{1}{2})}{2^{4(n+m)}\; t'^{\,2(n+m)}}\, P_{n+m,\nu}(y;-t')\right].$$

Properties of $G(x,y;t)$ are discussed in detail in [3].

The basic property of solutions of (1.1) central to our theory is given in the following definition.

<u>Definition 2.3</u>: A C^2 solution $u(x,t)$ of (1.1) such that

$$u(x,t) = \int_0^\infty G(x,y;t-t')\, u(y,t')\, d\mu(y) , \tag{2.4}$$

where

$$d\mu(x) = \frac{2^{\frac{1}{2}-\nu}}{\Gamma(\nu+\frac{1}{2})}\, x^{2\nu}\, dx ,$$

the integral converging absolutely for every t, t', $a < t' < t < b$, is said to have the Huygens property for $a < t < b$. We denote by H^* the class of such functions.

We describe next a subclass of even entire functions important in this study.

<u>Definition 2.4</u>: An even entire function

$$\varphi(x) = \sum_{n=0}^{\infty} a_n x^{2n} \tag{2.5}$$

belongs to the class (ρ, τ), or has growth (ρ, τ) if and only if

$$\overline{\lim_{n\to\infty}} \frac{n}{e\rho} |a_n|^{\rho/n} \leq \tau . \tag{2.6}$$

Note that this notation refers to $\varphi(x)$ as a function of x^2. The equivalent customary notation would be $(2\rho,\tau)$ for the growth of $\varphi(x)$ considered as a function of x.

3. <u>Region of convergence</u>:

We determine here the region of convergence of the Maclaurin series

$$(3.1)\qquad \sum_{n=0}^{\infty} \frac{x^{2n}}{n!\,\Gamma(\nu+\frac{1}{2}+n)} \sum_{m=0}^{\infty} a_{n+m} \frac{2^{2m} t^m}{m!} .$$

LEMMA 3.1. If

$$(3.2)\qquad \overline{\lim_{n\to\infty}} \frac{|a_n|^{1/n}}{n} = \frac{1}{4\sigma e} > 0 ,$$

then the series (3.1) converges for $|t| < \sigma$ and diverges for $|t| > \sigma$.

<u>Proof</u>: For any θ, $0 < \theta < 1$, we have, from (3.2)

$$|a_n| < K\left(\frac{n}{4\sigma e\theta}\right)^n$$

for some constant K. Hence

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!\,\Gamma(\nu+\frac{1}{2}+n)} \sum_{m=0}^{\infty} |a_{n+m}| \frac{2^{2m}|t|^m}{m!}$$

$$\leq K \sum_{n=0}^{\infty} \frac{x^{2n}}{n!\,\Gamma(\nu+\frac{1}{2}+n)} \sum_{m=0}^{\infty} \left(\frac{n+m}{4\sigma e\theta}\right)^{n+m} \frac{2^{2m}|t|^m}{m!}$$

$$= K \sum_{m=0}^{\infty} \frac{2^{2m}|t|^m}{m!} \sum_{n=m}^{\infty} \frac{x^{2n-2m}}{(n-m)!\,\Gamma(\nu+\frac{1}{2}+n-m)} \left(\frac{n}{4\sigma e\theta}\right)^n$$

$$= K \sum_{n=0}^{\infty} \left(\frac{n}{4\sigma e\Theta}\right)^n \frac{1}{n!\,\Gamma(\nu+\frac{1}{2}+n)} \sum_{m=0}^{n} 2^{2m}\binom{n}{m} \frac{\Gamma(\nu+\frac{1}{2}+n)}{\Gamma(\nu+\frac{1}{2}+n-m)}\, x^{2n-2m}\, |t|^m$$

$$= K \sum_{n=0}^{\infty} \left(\frac{n}{4\sigma e\Theta}\right)^n \frac{1}{n!\,\Gamma(\nu+\frac{1}{2}+n)} P_{n,\nu} \quad (x, |t|).$$

An appeal to lemma 3.3 of [6] yields, for any $\delta > 0$, the dominating series

$$\sum_{n=0}^{\infty} \left(\frac{n}{4\sigma e\Theta}\right)^n \frac{1}{n!\,\Gamma(\nu+\frac{1}{2}+n)} \left(1 + \frac{|t|}{\delta}\right)^{\nu+\frac{1}{2}} \left[\frac{4n(|t|+\delta)}{e}\right]^n e^{\frac{x^2}{4\delta}} ,$$

which converges for $|t| + \delta < \sigma\Theta$. By taking δ arbitarily close to zero and Θ near 1, we have established the absolute convergence of the series (3.1) for $|t| < \sigma$.

If the series (3.1) were to converge for some point (x_o, t_o), with $|t_o| > \sigma$, then, in particular, the simple series

$$(3.3) \qquad \sum_{m=0}^{\infty} a_m \frac{2^{2m}}{m!} t_o^m$$

must converge, and

$$(3.4) \qquad \overline{\lim_{n\to\infty}} \left[\frac{2^{2n}}{n!} |a_n|\right]^{1/n} \leq \frac{1}{|t_o|} ,$$

or, by Stirling's formula,

$$(3.5) \qquad \overline{\lim_{n\to\infty}} \frac{|a_n|^{1/n}}{n} \leq \frac{1}{4e|t_o|} .$$

Since $|t_o| > \sigma$, this contradicts hypothesis (3.2).

4. Series expansion:

We now establish our principal result.

THEOREM 4.1. A solution $u(x,t)$ of the generalized heat equation (1.1) has the Maclaurin expansion

$$(4.1) \qquad u(x,t) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!\Gamma(\nu+\frac{1}{2}+n)} \sum_{m=0}^{\infty} a_{n+m} \frac{2^{2m}\, t^m}{m!} ,$$

with

$$(4.2) \qquad a_k = \frac{k!\,\Gamma(\nu+\frac{1}{2}+k)}{(2k)!} u^{(2k)}(o,o) ,$$

for $|t| < \sigma$, $|x| < \sigma$, if and only if $u(x,t) \in H^*$ for $|t| < \sigma$, $0 \leq x < \infty$.

Proof: To prove sufficiency, we assume that $u(x,t) \in H^*$ for $|t| < \sigma$. Then, for t,t', $-\sigma < -t' < t < \sigma$, $0 < x < \infty$, by definition 2.3,

$$(4.3) \qquad u(x,t) = \int_0^{\infty} G(x,y;\, t+t')\, u(y,-t')\, d\mu(y) .$$

with the integral converging absolutely. We choose $t' > 0$. Now, substituting for $G(x,y;\, t+t')$ its series expansion (2.3), we have, provided termwise integration is valid,

$$(4.4) \quad u(x,t) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!\Gamma(\nu+\frac{1}{2}+n)} \sum_{m=0}^{\infty} \frac{2^{2m}t^m}{m!} \frac{\Gamma(\nu+\frac{1}{2})}{2^{4(n+m)}t'^{2(n+m)}} \times$$

$$\int_0^{\infty} G(y;t')P_{n+m,\nu}(y,-t')u(y,-t')d\mu(y) .$$

The validity of this result may be verified by an appeal to lemmas 3.4 and 4.7 of [6]. Hence

$$(4.5) \qquad u(x,t) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!\Gamma(\nu+\frac{1}{2}+n)} \sum_{m=0}^{\infty} a_{n+m} \frac{2^{2m}\, t^m}{m!} ,$$

where

$$(4.6) \quad a_k = \frac{\Gamma(\nu+\frac{1}{2})}{2^{4k}t'^{2k}} \int_0^{\infty} G(y;t')P_{k,\nu}(y,-t')\, u(y,-t')d\mu(y) ,$$

with the series converging for $|t| < \sigma$.

Since $u(x,t)$ as given by (4.3) is an analytic function of x, as established in theorem 3.2 of [4], we differentiate $u(x,o)$, with respect to x successively $2n$ times, taking the derivative under the integral sign.

Then, applying lemma 2.1 and comparing the result with (4.6), we have the determination (4.2) for the coefficients a_k .

Conversely, to prove the necessity of the conditions, assume that a solution $u(x,t)$ has the series expansion (4.1) for $|x| < \sigma$, $|t| < \sigma$. Now, for t, t', $-\sigma < t' < t < \sigma$, we have, by lemma 2.1 of [6],

$$\int_0^\infty G(x,y;\, t-t')\, u(y,t')\, d\mu(y)$$

$$= \sum_{n=0}^{\infty} \frac{P_{n,\nu}(x,t-t')}{n!\,\Gamma(\nu+\frac{1}{2}+n)} \sum_{m=0}^{\infty} a_{n+m} \frac{2^{2m}\, t'^{m}}{m!} ,$$

where termwise integration may be justified by lemma 3.1 and by lemma 3.3 of [6]. Using definition (1.2), we have

$$\int_0^\infty G(x,y;t-t')\, u(y,t')\, d\mu(y) =$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{\infty} a_{n+m} \frac{2^{2m} t'^{m}}{m!} \sum_{k=0}^{n} 2^{2k} \binom{n}{k} \frac{1}{\Gamma(\nu+\frac{1}{2}+n-k)}\, x^{2n-2k} (t-t')^{k}$$

$$= \sum_{m=0}^{\infty} \frac{2^{2m} t'^{m}}{m!} \sum_{k=0}^{\infty} \frac{2^{2k}}{k!} (t-t')^{k} \sum_{n=0}^{\infty} \frac{a_{n+m+k}}{n!\,\Gamma(\nu+\frac{1}{2}+n)}\, x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!\Gamma(\nu+\frac{1}{2}+n)} \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} a_{n+m} \frac{2^{2m}}{m!} \binom{m}{k} t'^{m-k} (t-t')^k$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!\Gamma(\nu+\frac{1}{2}+n)} \sum_{m=0}^{\infty} a_{n+m} \frac{2^{2m}}{m!} \sum_{k=0}^{m} \binom{m}{k} t'^{m-k} (t-t')^k$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!\Gamma(\nu+\frac{1}{2}+n)} \sum_{m=0}^{\infty} a_{n+m} \frac{2^{2m}}{m!} t^m$$

$= u(x,t)$.

Changes in order of summation at each step are verifiable. Hence, by definition 2.3, $u(x,t) \in H^*$ for $|t| < \sigma$, $|x| < \sigma$. By theorem 3.1 of [4] $u(x,t) \in H^*$ for $|t| < \sigma$, $0 \leq x < \infty$, and the proof is complete.

An example illustrating the theorem is given by $u(x,t) = e^{a^2 t} \mathcal{J}(ax)$ which belongs to H^* for $-\infty < t < \infty$. Here we have, for $-\infty < t < \infty$,

$$e^{a^2 t} \mathcal{J}(ax) = \Gamma(\nu+\tfrac{1}{2}) \sum_{n=0}^{\infty} \frac{(ax)^{2n}}{2^{2n} n!\Gamma(\nu+\frac{1}{2}+n)} \sum_{m=0}^{\infty} \frac{(a^2 t)^m}{m!} .$$

5. Simple series expansions in t:

Summing the double series (4.1) by rows, we may represent a generalized temperature function with the Huygens property by a simple Maclaurin series in t , as stated in the following result.

THEOREM 5.1. If $u(x,t) \in H^*$ for $|t| < \sigma$ and $f(x) = u(x,o)$, then

$$(5.1) \qquad u(x,t) = \sum_{k=o}^{\infty} \Delta_x^k f(x) \frac{t^k}{k!}, \qquad |t| < \sigma \quad ,$$

where

$$\Delta_x f(x) = f''(x) + \frac{2\nu}{x} f'(x) .$$

$f(x)$ is an even entire function of growth $(1, \frac{1}{4\sigma})$.

Proof: By theorem 4.1, we have

$$(5.2) \qquad u(x,t) = \sum_{k=o}^{\infty} \frac{2^{2k} t^k}{k!} \sum_{n=o}^{\infty} \frac{a_{n+k} x^{2n}}{n!\,\Gamma(\nu+\frac{1}{2}+n)} ,$$

so that

$$(5.3) \qquad f(x) = u(x,o)$$

$$= \sum_{n=o}^{\infty} \frac{a_n x^{2n}}{n!\,\Gamma(\nu+\frac{1}{2}+n)}, \qquad |x| < \infty .$$

Now, successive application of the operator Δ_x gives

$$\Delta_x^k f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!\Gamma(\nu+\frac{1}{2}+n)} \Delta_x^k x^{2n}$$

$$(5.4) \qquad = \sum_{n=k}^{\infty} \frac{a_n}{n!\Gamma(\nu+\frac{1}{2}+n)} 2^{2k} k!\binom{n}{k} \frac{\Gamma(\nu+\frac{1}{2}+n)}{\Gamma(\nu+\frac{1}{2}+n-k)} x^{2n-2k}$$

$$= 2^{2k} \sum_{n=0}^{\infty} \frac{a_{n+k}\, x^{2n}}{n!\Gamma(\nu+\frac{1}{2}+n)} .$$

Substituting (5.4) in (5.2), we find that

$$(5.5) \qquad u(x,t) = \sum_{k=0}^{\infty} \Delta_x^k f(x) \frac{t^k}{k!} .$$

Now, $f(x)$ is an even entire function. Further, by Stirling's formula, and by (3.5), we note that

$$(5.6) \qquad \overline{\lim_{n\to\infty}} \frac{n}{e} \left[\frac{|a_n|}{n!\Gamma(\nu+\frac{1}{2}+n)}\right]^{1/n} = \overline{\lim_{n\to\infty}} \frac{e\,|a_n|^{1/n}}{n}$$

$$< \frac{1}{4\sigma} .$$

Hence, it follows from definition 2.4 that $f(x)$ is of growth $(1, \frac{1}{4\sigma})$.

COROLLARY 5.2. If $u(x,t) \in H^*$ for $|t| < \sigma$, then $u(x,t_o)$, $|t_o| < \sigma$ is an even entire function of growth $(1, \frac{1}{4(\sigma - |t_o|)})$.

<u>Proof</u>: A Maclaurin expansion of $u(x,t)$ valid for $|t| < \sigma$ implies a Taylor expansion about $(0,t_o)$ valid for $|t-t_o| < \sigma - |t_o|$. If we set $t-t_o = t^*$ in the latter series, then by theorem 4.1, $u(x,t^*+t_o) \in H^*$ for $|t^*| < \sigma - |t_o|$ and an application of theorem 5.1 establishes the corollary.

COROLLARY 5.3. There exists a generalized temperature function $u(x,t)$ which is equal to its Maclaurin series in the strip $|t| < \sigma$ and which reduces to $f(x)$ for $t = 0$ if and only if $f(x)$ is an even entire function of growth $(1, \frac{1}{4\sigma})$.

<u>Proof</u>: That the condition is necessary is established by theorem 5.1. To prove sufficiency, we assume that $f(x)$ is an even entire function of growth $(1, \frac{1}{4\sigma})$ given by

$$(5.7) \qquad f(x) = \sum_{n=0}^{\infty} a_n x^{2n} .$$

Then, as in (5.4)

$$(5.8) \qquad \Delta_x^k f(x) = 2^{2k} \sum_{n=0}^{\infty} a_{n+k} \frac{(n+k)!}{n!} \frac{\Gamma(\nu+\frac{1}{2}+n+k)}{\Gamma(\nu+\frac{1}{2}+n)} x^{2n} .$$

Now consider the series

$$(5.9) \qquad \sum_{k=0}^{\infty} \Delta_x^k f(x) \frac{t^k}{k!} .$$

Since $f(x)$ is of growth $(1, \frac{1}{4\sigma})$, we have

$$(5.10) \qquad \overline{\lim_{n\to\infty}} \frac{n\,|a_n|^{1/n}}{e} \leq \frac{1}{4\sigma}$$

and consequently the series (5.9) may be shown to converge absolutely for $|t| < \sigma$, and represents there the generalized temperature function $u(x,t)$ sought. Clearly $u(x,0) = f(x)$.

Note that the function

$$f(x) = G(x;i)$$

$$= \left(\frac{1}{2i}\right)^{\nu+\frac{1}{2}} e^{\frac{x^2}{4}i}$$

is an even entire function of growth $(1, \frac{1}{4})$. As corollary 5.3 predicts, there exists a generalized temperature function $u(x,t) = G(x;\ t+i)$ where

$$G(x;t+i) = \left(\frac{1}{2i}\right)^{\nu+\frac{1}{2}} \sum_{k=0}^{\infty} \frac{2^{2k}t^k}{k!} \sum_{n=0}^{\infty} \frac{(i)^{n+k}}{n!\,2^{2(n+k)}} \frac{\Gamma(\nu+\frac{1}{2}+n+k)}{\Gamma(\nu+\frac{1}{2}+n)}\, x^{2n}\,,$$

$$|t| < 1\,.$$

6. Simple series expansions in x^2:

If the double series (4.1) is summed by columns, a generalized temperature function with the Huygens property may be represented by a simple Maclaurin series in x^2.

THEOREM 6.1. If $u(x,t) \in H^*$ for $|t| < \sigma$ and $g(t) = u(0,t)$, then

$$(6.1)\quad u(x,t) = \sum_{n=0}^{\infty} g^{(n)}(t)\, \frac{\Gamma(\nu+\frac{1}{2})}{2^{2n}\, n!\,\Gamma(\nu+\frac{1}{2}+n)}\, x^{2n}\,, \quad |t| < \sigma\,.$$

Proof: By theorem 4.1, we have

$$(6.2)\quad u(x,t) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!\,\Gamma(\nu+\frac{1}{2}+n)} \sum_{k=0}^{\infty} a_{n+k}\, \frac{2^{2k}t^k}{k!}\,, \quad |t| < \sigma\,,$$

so that

$$(6.3)\qquad g(t) = u(0,t)$$

$$= \frac{1}{\Gamma(\nu+\frac{1}{2})} \sum_{k=0}^{\infty} a_k\, \frac{2^{2k}t^k}{k!}$$

Successive differentiation of (6.3) yields

$$g^{(n)}(t) = \frac{2^{2n}}{\Gamma(\nu+\frac{1}{2})} \sum_{k=0}^{\infty} a_{n+k} \frac{2^{2k} t^k}{k!} . \tag{6.4}$$

Substituting (6.4) in (6.2), we derive (6.1) as required.

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